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The motion of a body with a plane of symmetry over a three-dimensional sphere under the action of a spherical analogue of Newtonian gravitation☆

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ABSTRACT

The problem of the motion of a rigid body possessing a plane of symmetry over the surface of a three-dimensional sphere under the action of a spherical analogue of Newtonian gravitation forces is considered. Approaches to introducing spherical analogues of the concepts of centre of mass and centre of gravity are discussed. The spherical analogue of "satellite approach" in the problem of the motion of a rigid body in a central field, which arises on the assumption that the dimensions of the body are small compared with the distance to the gravitating centre, is studied. Within the framework of satellite approach, assuming plane motion of the body, the question of the existence and stability of steady motions is investigated. A spherical analogue of the equation of the plane oscillations of a body in an elliptic orbit is derived.

The study of the mechanics of systems in motion on the surface of a sphere and a pseudosphere, dating back to the investigations by Lobachevskii and Bolyai, has attracted the attention of many researchers. In particular, spherical and pseudospherical analogues of a number of classical problems of the mechanics of a point, such as the Bertrand–Kepler problem and the problem of two gravitating centres, have been studied (see, for example, the review by Dombrowski and Zitterbarth[1] with a subsequent addendum by Shchepetilov,[2] and also the earlier collection edited by Borisov and Mamayev[3]). In the course of more recent investigations,[4–8] the main classical results were corroborated. In these studies, explicit integration of the equations of motion was also carried out, and analogues of Kepler's laws were formulated and substantiated. Besides the studies collected in the well-known review edited by Borisov and Mamayev,[3] among recent investigations we will single out Ref. 8 in which "action–angle" variables were introduced for potentials ensuring the solution of Bertrand's problem, which made it possible to move on to an investigation of quantum analogues of the problems under consideration. We also mention investigations devoted to the non-integrability of the restricted two-body problem[9] and to quantization in the two-body problem.[10,11]

Investigations of rigid body dynamics in non-Euclidean spaces, dating back, it appears, to the work of Killing and Zhukovskii,[12,13] with the prime aim of deriving equations of motion and the correct introduction of the concept of the centre of mass, were continued in Refs. 14–19 (see also Refs. 20–22). The qualitative behaviour of an axisymmetric top was investigated in Ref. 23. The existence, stability and bifurcation of the steady motions of a dumb-bell-shaped body in a central gravitation field have also been studied.[24,25]

1. Formulation of the problem

Suppose that in a four-dimensional Euclidean space R^4 with a fixed absolute system of coordinates $OX_1X_2X_3X_4$ a three-dimensional sphere of unit radius, with centre at the origin of coordinates, the point O, is embedded in the standard way and is given by the equation

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 = 1 \tag{1.1}$$

Besides the sphere (1.1), we will consider its plane section, a two-dimensional sphere, given by the relation

$$X_1^2 + X_2^2 + X_3^2 = 1, \quad X_4 = 0 \tag{1.2}$$

Let $\mathcal{G}$ be the rigid body inserted into the sphere (1.1). As this sphere allows of a group of isometries, the body $\mathcal{G}$ can rotate freely within it. Let C be a point fixed in the body, and $Cx_1x_2x_3x_4$ be a right-handed orthogonal system of coordinates connected to the body such that the Cx_3 axis is directed from the centre of the sphere along its radius. Then the Cx_1, Cx_2 and Cx_3 axes are positioned in a tangential plane to the sphere (1.1) at the point C. For convenience, it is also possible to consider the system of coordinates $Ox_1x_2x_3x_4$

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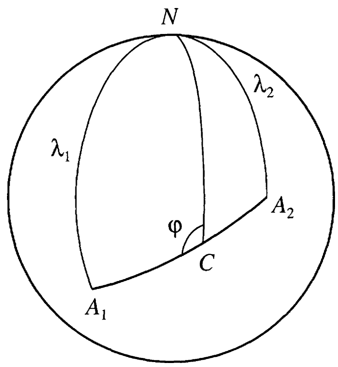

Fig. 1.

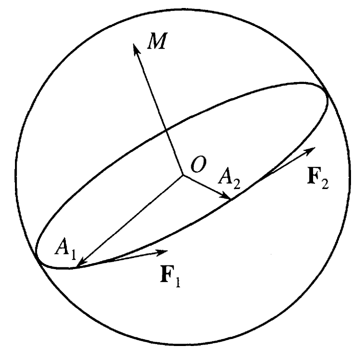

Fig. 2.

connected to the body, the axes of which are parallel to the corresponding axes of the system $Cx_1x_2x_3x_4$, while the origin coincides with the centre of the sphere.

Let us assume that the distribution of the masses of the body allows of a plane of symmetry $x_4 = 0$. This occurs, for example, in the case where all points of the body are concentrated in the indicated plane. Let us assume that, at the initial instant of time, the planes $X_4 = 0$ and $x_4 = 0$ coincide, and the projections of the vectors of the velocities of all points forming the body onto the normal to this plane are zero. Then, these two planes coincide for the entire time of motion, and the section of the body with the plane $x_4 = 0$, denoted by $\mathcal{G}_2$, remains on the sphere (1.2) for the entire time of motion. Such motions are the main subject of the present paper.

The spherical plate $\mathcal{G}_2$ can be considered as a rigid body rotating about the point O in the plane absolute space R^3 formed by the axes $OX_1X_2X_3$. The massive points A_i ($i = 1, 2, \ldots$) forming the plate are specified by their projections $\mathbf{r}_i = (r_{i1}, r_{i2}, r_{i3})$ onto the axes of the system of coordinates $OX_1X_2X_3$ connected to the body, remaining constrained by the unique relation

$$r_{i1}^2 + r_{i2}^2 + r_{i3}^2 = 1$$

which indicates that the points are situated on a sphere of unit radius.

The following spherical coordinates, introduced with respect to the axes $Ox_1x_2x_3$, will prove useful

$$r_{1i} = \sin\lambda_i\cos\varphi_i, \quad r_{2i} = \sin\lambda_i\sin\varphi_i, \quad r_{3i} = \cos\lambda_i \qquad (1.3)$$

The angles λ_i specify the polar radii of the points and are equal to the angular distance between the Cx_3 axis and the vector OA_i. The angles $\pi/2 - \lambda_i$ are normally referred to as the latitudes. The polar angles φ_i (longitudes) are specified by the angles between the COX_1 and COA_i planes (Fig. 1).

2. Geometrical statics

2.1. The concept of force

To formulate the principal positions of spherical geometrical statics, and to compare them with the analogous propositions of statics in the case of a plane space, it is useful to introduce the spherical analogue of the concept of force and investigate its properties. For convenience, we will dwell on the two-dimensional case – in the general case the situation will have a similar nature. In this case, we will actively use the above-mentioned analogue of the problem of the motion of a body over a sphere and the problem of the motion of a rigid body about a stationary point.

Let point A from the sphere (1.2) be subjected to a certain (force) action, making it change position. From the viewpoint of the indicated analogue, a moment $\mathbf{M}$ that applied to the rigid body OA forces it to rotate about the stationary point O. If the moment $\mathbf{M}$ is perpendicular to the segment OA, it can be represented in the form

$$\mathbf{M} = OA \times \mathbf{F} \qquad (2.1)$$

where the vector $\mathbf{F}$ is naturally considered to be the force applied to the point A. Considering relation (2.1) to be the equation relative to the vector $\mathbf{F}$ and bearing in mind the uniqueness of the radius of the sphere (1.2) and the perpendicularity of the vectors $\mathbf{M}$ and OA, we will write the solution of this equation as

$$\mathbf{F} = \mathbf{M} \times OA \qquad (2.2)$$

The force $\mathbf{F}$ defined in this way is perpendicular both to the vector $\mathbf{M}$ and to the vector OA, and is situated in a plane tangential to the sphere (1.2) at the point A.

As is well-known, force is considered to be a sliding vector in the mechanics of flat space. On a two-dimensional sphere, the situation is analogous: using the concept of parallel transfer along a curve, we will transfer the force along its line of action, which is defined by it as a great-circle tangential vector. With such a method of force transfer, the moment vector $\mathbf{M}$ from relation (2.1) remains unchanged (Fig. 2):

$$\mathbf{M} = OA_1 \times \mathbf{F}_1 = OA_2 \times \mathbf{F}_2, \quad \mathbf{F}_i = \mathbf{F}(A_i)$$

Here and below, the argument in the expressions for the forces denotes their point of application.

2.2. The composition of forces

Suppose that forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are applied at the points A_1 and A_2 of the body $\mathcal{G}_2$. As on a two-dimensional sphere, any two great circles intersect, the lines of action of these forces are sure to intersect (or concide). The case of coincidence is simple: the composition of these forces reduces to their transfer to any point of the circle – their line of action – and to subsequent composition as vectors positioned on one line. In the case of the intersection of the lines of action of the forces, we will transfer these forces to either of two points of intersection of their lines of action. These forces, now specifying a common plane tangential to the sphere, then combine according to the parallelogram law. The line of action of the resulting force also passes through the indicated points of intersection of the circles (Fig. 3).

If α is the angle between these forces, referred to the common point A, and f_1 and f_2 are their magnitudes, then the magnitude of the resulting force is given by the expression

$$f = (f_1^2 + 2f_1f_2\cos\alpha + f_2^2)^{1/2} \qquad (2.3)$$

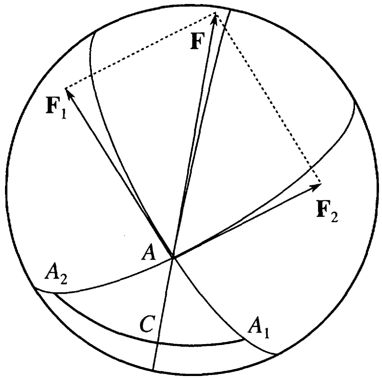

Fig. 3.

Here, the resulting force, referred to the point A, makes with $\mathbf{F}_1$ and $\mathbf{F}_2$ angles α_1 and α_2 such that

$$f_1 \sin\alpha_1 = f_2 \sin\alpha_2, \quad \alpha = \alpha_1 + \alpha_2 \tag{2.4}$$

which is obtained, for example, as a result of applying the sine theorem to any of the triangles comprising the parallelogram of forces in Fig. 3.

Let us now consider, from the viewpoint of geometrical statics, the spherical analogue of the lever law. Suppose the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are applied to the ends of a lever – points A_1 and A_2 respectively. In other words, these forces generate a moment rotating the rigid body OA_1A_2 about the stationary point O. If A is one of the points of intersection of the lines of action of these forces, then, transferring the force to this point, we have

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 = OA_1 \times \mathbf{F}_1(A_1) + OA_2 \times \mathbf{F}_2(A_2) =$$
$$= OA \times \mathbf{F}_1(A) + OA \times \mathbf{F}_2(A) = OA \times \mathbf{F}(A)$$
$$\mathbf{F}(A) = \mathbf{F}_1(A) + \mathbf{F}_2(A)$$

For the lever to remain in equilibrium, a force $\mathbf{F}(A)$ must be applied to it or, what amounts to the same thing, a moment $\mathbf{M}$ must be applied to the body.

Relation (2.4) is naturally considered to be the spherical analogue of the lever law. The point C of intersection of the line of action of the resulting force $\mathbf{F}$ and the lever A_1A_2, provided with a mass f defined by relation (2.4), is naturally considered to be the centre of mass of the system of points A_1 and A_2 with masses f_1 and f_2.

3. Analytical statics

Let us assume that at the point N on the surface of the sphere (1.1) there is a gravitating centre. As is well known, the analogue of Newtonian gravitational potential for a three-dimensional sphere, determined from the solution of the Laplace–Beltrami equation, is proportional to the cotangent of the angle between two interacting points. If in the axes connected to the body the single vector ON is given as $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$, then the interaction potential is written as

$$U = -G\sum_i \mu_i \frac{\cos\theta_i}{\sin\theta_i}, \quad \mu_i > 0$$
$$\cos\theta_i = \gamma_1 r_{1i} + \gamma_2 r_{2i} + \gamma_3 r_{3i}, \quad \sin\theta_i = (1 - \cos^2\theta_i)^{1/2} \tag{3.1}$$

where G is the product of the "gravitational constant" and the mass of the gravitating centre.

Let us assume that the body is small compared with the radius of the sphere (1.1), i.e.,

$$|r_{1i}| \ll 1, \quad |r_{2i}| \ll 1$$

and is positioned close to the point C of intersection of its third axis of inertia and the surface of the sphere (1.2). Then, bearing in mind the relation

$$r_{3i} = (1 - r_{1i}^2 - r_{2i})^{1/2}$$

we will write the expansion of the potential with respect to the corresponding small parameter in the form

$$U = U_0 + U_1 + U_2 + \dots$$

$$U_0 = -GM\frac{\gamma_3}{(1-\gamma_3^2)^{1/2}}, \quad M = \sum_i \mu_i$$

$$U_1 = -\frac{\Lambda_1\gamma_1 + \Lambda_2\gamma_2}{(1-\gamma_3^2)^{3/2}}, \quad \Lambda_k = \sum_i \mu_i r_{ki}, \quad k = 1, 2 \quad U_2 =$$
$$-G\frac{\gamma_3[(P_{11}+2P_{12}+P_{22})(\gamma_3^2-1) + 3(P_{11}\gamma_1^2+2P_{12}\gamma_1\gamma_2+P_{22}\gamma_2^2)]}{2(1-\gamma_3^2)^{5/2}}$$
$$P_{kl} = \sum_i \mu_i r_{ki} r_{li}, \quad k, l = 1, 2, 3 \tag{3.2}$$

where P_{kl} are components of the Poinsot tensor. From now on we will assume that $P_{ij} = 0$, $i \neq j$, which can be achieved by appropriate choice of the system of coordinates connected to the body. Note also that

$$M = P_{11} + P_{22} + P_{33}$$

If the point C is selected in such a way that

$$\Lambda_1 = \Lambda_2 = 0 \tag{3.3}$$

then the expansion of the potential begins with terms of the second order of smallness. In other words, the point C, selected in this way, is naturally understood to be the *centre of gravity* of the body: in this case, if the body is fastened at the point C, then, as a first approximation, a neutral position of equilibrium occurs.

Using relation (3.3), we will try to find the spherical analogue of Archimedes' lever law. We will assume that the lever is formed by the points A_1 and A_2, with masses μ_1 and μ_2, connected by a segment of the great arc of a two-dimensional sphere. We will use relations (1.3), in which we will assume that $\varphi = 0$. Then, by virtue of the first relation of (3.3),

$$\mu_1 \sin\lambda_1 = \mu_2 \sin\lambda_2 \tag{3.4}$$

and the second relation is satisfied identically. Relation (3.4) expresses the spherical analogue of the lever law and can be considered to be the definition of the spherical analogue of the centre of gravity.

Remark 1. The centre of gravity concept introduced in this way is in agreement with the centre of mass concept introduced elsewhere[21] on the basis of formal mathematical axioms (cf. Ref. 26):

1. The centre of mass of a single-point system is the point itself.
2. The centre of mass of the centres of mass of two systems of point masses coincides with the centre of mass of the combination of points of these systems.
3. The multiplication of all masses of a system of point masses by the same number does not alter the position of the centre of mass but does entail the multiplication of the overall mass by the same number.
4. The centre of mass is invariant under the displacements of the system of point masses as a rigid whole.
5. The position of the centre of mass is continuous in the natural topology of the system of point masses.

This cannot be said of the centre of mass concept based on the geometrical statics considerations introduced above.

To investigate the existence and stability of equilibria of a lever in the case where condition (3.4) is satisfied, we will employ, as usual, the Euler angles

$$\gamma_1 = \sin\theta\sin\varphi, \quad \gamma_2 = \sin\theta\cos\varphi, \quad \gamma_3 = \cos\theta$$

In these angles, the potential U_2, apart from an additive constant that is independent of the angle ϕ, has the form

$$U_2 = -G\frac{3}{2}\frac{\cos\theta}{\sin^3\theta}(P_{11}\sin^2\varphi + P_{22}\cos^2\varphi) =$$
$$= -G\frac{3}{2}\frac{\cos\theta}{\sin^3\theta}(\mu_1\sin^2\lambda_1 + \mu_2\cos^2\lambda_2)\sin^2\varphi + \text{const}$$

In this case there are four equilibria, which are determined from the equation

$$\frac{\partial U_2}{\partial\varphi} = -3G\sin\varphi\cos\varphi\frac{\cos\theta}{\sin^3\theta}(\mu_1\sin^2\lambda_1 + \mu_2\cos^2\lambda_2) = 0$$

These are the horizontal equilibria $\varphi = 0$ and $\varphi = \pi$, and the vertical equilibria $\varphi = \pi/2$ and $\varphi = 3\pi/2$. Sufficient conditions for their stability are defined by the inequality

$$\frac{\partial^2 U_2}{\partial\varphi^2} = -3G(\cos^2\varphi - \sin^2\varphi)\frac{\cos\theta}{\sin^3\theta}(\mu_1\sin^2\lambda_1 + \mu_2\cos^2\lambda_2) > 0$$

It can be seen that the two vertical and the two horizontal equilibria possess the same stability property. Moreover, if the vertical equilibria are stable, then the horizontal equilibria are unstable, and vice versa.

It is remarkable that, unlike the plane case, the stability condition depends principally on the position of the suspension point. If the suspension point is in the "northern hemisphere" (N is the "North Pole"), then the vertical equilibria are stable and the horizontal equilibria are unstable. If the suspension point is in the southerm hemisphere, then, conversely, the horizontal equilibria are stable and the vertical equilibria are unstable.

If the suspension point is on the equator, then, as a second approximation, again there is a continuous family of neutral equilibria, and, in order to investigate the existence of equilibria and their stability, it is necessary to take into account terms of the third-order smallness in the expansion of the potential.

Remark 2. In the region adjacent to the equator, where the quantities r_{1i}, r_{2i} and γ_3 have the same order of smallness, instead of expansion (3.2), generally speaking, it is better to use expansion in the small parameter that arises and which has the form

$$U = U'_0 + U'_1 + U'_2 + U'_3 + \dots$$

$$U'_0 = 0, \quad U'_1 = G(\gamma_1\Lambda_1 + \gamma_2\Lambda_2 + M\gamma_3), \quad U'_2 = 0$$

$$U'_3 = \frac{G}{2}\sum_i[-\gamma_3(r_{1i} + r_{2i})^2 + (\gamma_1 r_{1i} + \gamma_2 r_{2i} + \gamma_3)^2]$$

In the natural case of a non-zero mass of the body, its motion is determined by terms of the first order of smallness.

In the precise formulation, by virtue of symmetry, there are two vertical equilibria in the problem. However, unlike the flat case, in the problem being considered there are families of inclined equilibria "born" from the vertical equilibria when the parameters of the problem pass through critical values. In this case, as usual, there is a change in the stability properties of the vertical equilibria.

Remark 3. In flat space, the centre of mass concept can be introduced not only from static but also from dynamic considerations. In fact, the centre of mass is the only body-connected point moving uniformly and rectilinearly with any free motion of the body. On the other hand, the centre of mass is the only point of the rigid body belonging to any axis of its free permanent rotation. In plane space, both these concepts and the "statically defined" centre of mass concepts are identical.

However, the transfer of centre of mass concepts based on dynamic representations to the case of a sphere is difficult. In fact, a point of a body moving uniformly during free motion of the body along a great circle simply does not exist. At the same time, there are a minimum of six points belonging to the body that can be considered as centres of rotation of the body at an arbitrary angular velocity.

4. Dynamics

Consider the dynamics of a spherical plate as a rigid body moving about a stationary point coinciding with the centre of the sphere – the point O. The kinetic energy has the form

$$T = \frac{1}{2}(\mathbf{I}\boldsymbol{\omega}, \boldsymbol{\omega}) \tag{4.1}$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is the angular velocity vector, $\mathbf{I}$ is the inertia tensor of the body about the stationary point, and its components in the principal axes are

$$I_1 = \sum_i m_i(r_{2i}^2 + r_{3i}^2) = P_{22} + P_{33} \ (1\ 2\ 3) \tag{4.2}$$

The kinetic energy can be expressed in the same way as a function of the Euler angles and their time derivatives. In Euler angles, the coordinates of the angular velocity vector have the form

$$\omega_1 = \dot\psi\sin\theta\sin\varphi + \dot\theta\cos\varphi, \ \omega_2 = \dot\psi\sin\theta\cos\varphi - \dot\theta\sin\varphi,$$
$$\omega_3 = \dot\psi\cos\theta + \dot\varphi$$

The angle ψ is measured by the angular distance between the OX_1 axis and the line of intersection of the OX_1X_2 and Ox_1x_2 planes. Here, the kinetic energy acquires the form

$$T = \frac{1}{2}(I_1(\dot\psi\sin\theta\sin\varphi + \dot\theta\cos\varphi)^2 +$$
$$I_2(\dot\psi\sin\theta\cos\varphi - \dot\theta\sin\varphi)^2 + I_3(\dot\psi\cos\theta + \dot\varphi)^2)$$

The motion of the plate can be described using the Euler–Poisson equations

$$\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{\omega}} = \frac{\partial L}{\partial \boldsymbol{\omega}} \times \boldsymbol{\omega} + \frac{\partial L}{\partial \boldsymbol{\gamma}} \times \boldsymbol{\gamma}, \quad \frac{d\boldsymbol{\gamma}}{dt} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \quad L = T - U \tag{4.3}$$

or by means of the normal Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}, \quad x \in \{\psi, \theta, \varphi\} \tag{4.4}$$

Eqs. (4.3) and (4.4), apart from the energy integral

$$\mathcal{J}_0 = \left(\frac{\partial L}{\partial \boldsymbol{\omega}}, \boldsymbol{\omega}\right) - L = \sum_{x \in \{\psi, \theta, \varphi\}} \left(\frac{\partial L}{\partial \dot{x}}, \dot{x}\right) - L = T + U = h \tag{4.5}$$

allow of the area integral

$$\mathcal{J}_1 = \left(\frac{\partial L}{\partial \boldsymbol{\omega}}, \boldsymbol{\gamma}\right) = p_\psi \tag{4.6}$$

while Eq. (4.3) also allow of the geometric integral

$$\mathcal{J}_2 = (\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 1 \tag{4.7}$$

To integrate the equations of motion, in the general case one additional integral is lacking.

5. The satellite approximation

Let us consider the motion of a body on the assumption that its dimensions are small compared with the radius of the sphere (1.2). Again we will assume that the body is positioned close to the point C of intersection of its third axis of inertia and the surface of the sphere (1.2), and this point is the centre of gravity of the body.

We will represent Eq. (4.3) in the form

$$\begin{aligned}
(M - P_{11})\dot{\omega}_1 &= (M - P_{11} - 2P_{22})\omega_2\omega_3 - \\
&\quad GM\frac{\gamma_2}{1-\gamma_3^2} + \gamma_2\frac{\partial U_2}{\partial \gamma_3} - \gamma_3\frac{\partial U_2}{\partial \gamma_2} \\
(M - P_{22})\dot{\omega}_2 &= -(M - P_{22} - 2P_{11})\omega_3\omega_1 + \\
&\quad GM\frac{\gamma_1}{1-\gamma_3^2} + \gamma_3\frac{\partial U_2}{\partial \gamma_1} - \gamma_1\frac{\partial U_2}{\partial \gamma_3} \\
(P_{11} + P_{22})\dot{\omega}_3 &= (P_{11} - P_{22})\omega_1\omega_2 - G\frac{3(P_{22} - P_{11})\gamma_1\gamma_2}{(1-\gamma_3^2)^{5/2}}
\end{aligned} \tag{5.1}$$

Because, in the approximation under consideration, the following inequalities are satisfied

$$P_{11} \ll M, \quad P_{22} \ll M \tag{5.2}$$

system (5.1) can be represented in the form (cf., for example, Ref. 27)

$$\begin{aligned}
&\dot{\omega}_1 = \omega_2\omega_3 - G\frac{\gamma_2}{1-\gamma_3^2}, \quad \dot{\omega}_2 = -\omega_3\omega_1 + G\frac{\gamma_1}{1-\gamma_3^2}, \\
&\dot{\omega}_3 = K\omega_1\omega_2 - G\frac{3K\gamma_1\gamma_2}{(1-\gamma_3^2)^{5/2}} \\
&K = \frac{P_{11} - P_{22}}{P_{11} + P_{22}}
\end{aligned} \tag{5.3}$$

Eq. (5.3) must be supplemented by Poisson's equations.

Equations of this type were used earlier to describe restricted problems in rigid body mechanics.[27,28] These equations turn out to be partially integrable, i.e., the motion in certain degrees of freedom can be described in explicit form. However, unlike the problems considered in the publications mentioned, this circumstance is not so obvious in the problem under examination. Therefore, to integrate the equations of motion, we will use the Euler angles.

We will focus primarily on the fact that the expression for the kinetic energy has the form

$$\begin{aligned}
T &= T_0 + T_2 \\
T_0 &= \frac{1}{2}M(\omega_1^2 + \omega_2^2) = \frac{1}{2}M(\dot{\theta}^2 + \sin^2\theta\dot{\psi}^2) \\
T_2 &= \frac{1}{2}[P_{11}(\omega_3^2 - \omega_1^2) + P_{22}(\omega_3^2 - \omega_2^2)] = \\
&= \frac{1}{2}P_{11}((\dot{\psi}\cos\theta + \dot{\varphi})^2 - (\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi)^2) + \\
&+ P_{22}((\dot{\psi}\cos\theta + \dot{\varphi})^2 - (\dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi)^2)
\end{aligned}$$

Bearing in mind relations (5.2), the Lagrange equations of the second kind can be represented in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\psi}} = \frac{\partial L}{\partial \psi} \Rightarrow \frac{d}{dt}(\sin^2\theta\dot{\psi}) = 0 \tag{5.4}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \Rightarrow \frac{d}{dt}(\dot{\theta}) = \sin\theta\cos\theta\dot{\psi}^2 - \frac{G}{\sin^2\theta} \tag{5.5}$$

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial L}{\partial \varphi} \Rightarrow \frac{d}{dt}(\dot{\psi}\cos\theta + \dot{\varphi}) = \\
&= K\Big(-(\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi)(\dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi) \\
&\quad + \frac{3G\cos\theta\sin\varphi\cos\varphi}{\sin^3\theta}\Big)
\end{aligned} \tag{5.6}$$

Eqs. (5.4) and (5.5) are separated from Eq. (5.6). They describe the motion of the centre of gravity of the body – the point C. The corresponding quadratures are well known (see, for example, Ref. 7); in the notation adopted, they are obtained in the following way. From the first integral, occurring by virtue of Eq. (5.4),

$$\sin^2\theta\dot{\psi} = p_\psi \Rightarrow \dot{\psi} = \frac{p_\psi}{\sin^2\theta}$$

Substituting this expression into Eq. (5.5) we can represent the latter in the form

$$\ddot{\theta} = p_\psi^2\frac{\cos\theta}{\sin^3\theta} - \frac{G}{\sin^2\theta} \tag{5.7}$$

The first integral of this equation

$$\frac{1}{2}\left(\dot{\theta}^2 + \frac{p_\psi^2}{\sin^2\theta}\right) - G\frac{\cos\theta}{\sin\theta} = h \tag{5.8}$$

enables us to define the region in which the nutation angle θ varies

$$\frac{1}{2}\frac{p_\psi^2}{\sin^2\theta} - G\frac{\cos\theta}{\sin\theta} \le h$$

To integrate Eq. (5.7), with $p_\psi \neq 0$, as in the classical Kepler problem, we change to a new time, for which the spherical analogue of the true anomaly – the angle ψ – is used. In implicit form, the solution can be represented as (cf. Ref. 7)

$$\cos\theta = EP^{-1/2}, \quad \sin\theta = pP^{-1/2}, \quad E = 1 + e\cos\psi,$$

$$P = p^2 + E^2$$

$$p_\psi^2 = Gp, \quad e^2 = 1 + (2p_\psi^2 H - p_\psi^4)/G^2 \tag{5.9}$$

Here, the following relations hold

$$\dot\psi = G^{1/2} p^{-3/2} P^{1/2}, \quad \dot\theta = G^{1/2} p^{-1/2} e\sin\psi \tag{5.10}$$

We will introduce into Eq. (5.6) the true anomaly as a new independent variable. We have

$$\dot\psi \frac{d}{d\psi}\left[\dot\psi\left(\cos\theta + \frac{d\varphi}{d\psi}\right)\right] =$$

$$= -K(\dot\psi\sin\theta\sin\varphi + \dot\theta\cos\varphi)(\dot\psi\sin\theta\cos\varphi - \dot\theta\sin\varphi) + GK\frac{3\cos\theta\sin\varphi\cos\varphi}{\sin^3\theta}$$

Substitution of expressions (5.9) and (5.10) into this equation and subsequent simplification yields

$$P\frac{d}{d\psi}\left(EP^{1/2} + P\frac{d\varphi}{d\psi}\right) =$$

$$= -Kp^2(P^{1/2}\sin\varphi + e\sin\psi\cos\varphi)(P^{1/2}\cos\varphi - e\sin\psi\sin\varphi) + 3KEP\sin\varphi\cos\varphi \tag{5.11}$$

We have a spherical analogue of the equation of plane vibrations of a satellite in an elliptic orbit.[29]

In the case of the motion of the centre of mass in a circular orbit, the eccentricity e vanishes, and the equation being considered acquires the form

$$(p^2 + 1)\frac{d^2\varphi}{d\psi^2} = K(3 - p^2)\sin\varphi\cos\varphi \tag{5.12}$$

This equation is completely integrable. The energy integral has the form

$$\mathcal{J}_0 = \frac{1}{2}(p^2 + 1)\left(\frac{\partial\varphi}{\partial\psi}\right)^2 - \frac{K}{2}(3 - p^2)\sin^2\varphi$$

Remark 4. Using reasoning similar to the foregoing, it is possible to write the spherical analogue of the equations of spatial motions of a satellite about its centre of mass.

Remark 5. Eq. (5.11) are Lagrangian. The corresponding Lagrangian function has the form

$$\Lambda(\varphi', \varphi, \psi) = L_2(\dot\psi, \dot\theta, \varphi'\dot\psi, \theta, \varphi)/\dot\psi, \quad L_2(\dot\psi, \dot\theta, \dot\varphi, \theta, \varphi) = T_2 - U_2$$

where the quantities θ, $\dot\theta$ and $\dot\psi$ are replaced by functions of the true anomaly ψ, according to relations (5.10) and (5.11). By a canonical transformation we can write the equations of motion using Hamilton's equations.

6. The relative equilibria of a satellite in a circular orbit

We will investigate the features of the motion of a satellite about a centre of mass in a circular orbit. We will assume that $K > 0$, i.e., that the satellite is elongated along its first axis of inertia. According to Eq. (5.12), in the satellite there are two pairs of relative equilibria: 1) "tangential", in which $\varphi = 0$ and $\varphi = \pi$; 2) "radial", in which $\varphi = +\pi/2$.

If the following condition is satisfied

$$3 - p^2 > 0 \tag{6.1}$$

i.e., in the case of low and moderate values of the constant of the area integral, the radial relative equilibria are stable and the tangential relative equilibria are unstable.

When the inequality opposite to (6.1) is satisfied, the horizontal equilibria are stable, whereas the vertical equilibria are unstable. The latter effect is not observed for satellites moving in a flat space.

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